

On the exponent of a finite group admitting a fixed-point-free four-group of automorphisms

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ABSTRACT. Let A be a group isomorphic with either S_4 , the symmetric group on four symbols, or D_8 , the dihedral group of order 8. Let V be a normal four-subgroup of A and α an involution in $A \setminus V$. Suppose that A acts on a finite group G in such a manner that $C_G(V) = 1$ and $C_G(\alpha)$ has exponent e . We show that if $A \cong S_4$ then the exponent of G is e -bounded and if $A \cong D_8$ then the exponent of the derived group G' is e -bounded. This work was motivated by recent results on the exponent of a finite group admitting an action by a Frobenius group of automorphisms.

1. Introduction

Let G be a group admitting an action of a group A . We denote by $C_G(A)$ the set $C_G(A) = \{x \in G; x^a = x \text{ for any } a \in A\}$, the centralizer of A in G (the fixed-point group). In many cases the properties of $C_G(A)$ have influence over those of G . In particular, it was discovered in the late 90s that the exponent of $C_G(A)$ may have strong impact over the exponent of G [7]. Special attention was recently given to the situation where a Frobenius group acts by automorphisms on another group. Recall that a Frobenius group FH with kernel F and complement H can be characterized as a finite group that is a semidirect product of a normal subgroup F by H such that $C_F(h) = 1$ for every $h \in H \setminus \{1\}$. By Thompson's theorem [20] the kernel F is nilpotent, and by Higman's theorem [3] the nilpotency class of F is bounded in terms of the least prime divisor of $|H|$ (explicit upper bounds for the nilpotency class are due to Kreknin and Kostrikin [8, 9]). Suppose that

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the Frobenius group FH acts on a finite group G in such a way that $C_G(F) = 1$. It was shown in [5] that in this case the order and rank of G are bounded in terms of $|H|$ and the order and rank of $C_G(H)$, respectively. Further, it was shown that if F is cyclic, then the nilpotency class of G is bounded in terms of $|H|$ and the nilpotency class of $C_G(H)$. In the case when GF is also a Frobenius group with kernel G and complement F (so that GFH is a double Frobenius group) the latter result was obtained earlier in [13]. This solved in the affirmative Mazurov's problem 17.72(a) in Kourovka Notebook [6].

The other problem of Mazurov about double Frobenius groups – Problem 17.72(b) in Kourovka Notebook – is whether in a double Frobenius group GFH the exponent of G is bounded in terms of $|H|$ and the exponent of $C_G(H)$ only. That problem seems to be very hard and so far no viable approach to it has been found. We will quote just one result from [5] that indirectly addresses the problem:

THEOREM 1.1. *Suppose that a Frobenius group FH with cyclic kernel F and complement H acts on a finite group G in such a manner that $C_G(F) = 1$ and $C_G(H)$ has exponent e . Then the exponent of G is bounded solely in terms of e and $|FH|$.*

Since the exponent of G here depends on the order of F , the above theorem does not yield an answer to Mazurov's problem. The proof of Theorem 1.1 uses Lazard's Lie algebra associated with the Jennings–Zassenhaus filtration and its connection with powerful p -groups.

It is natural to ask if the theorem remains valid without assuming that F is cyclic. The case of the smallest Frobenius group whose kernel is non-cyclic was treated in [19]. The group in question is of course the non-abelian group of order 12 also known as the Alternating group A_4 of degree 4. The following theorem is the main result of [19].

THEOREM 1.2. *Suppose that the Frobenius group FH of order 12 acts coprimely on a finite group G in such a manner that $C_G(F) = 1$ and $C_G(H)$ has exponent e . Then the exponent of G is bounded in terms of e only.*

Recall that an action of a finite group A on a finite group G is coprime if $(|G|, |A|) = 1$. It is amazing that tools required for the treatment of the situation where FH has order 12 are more sophisticated than those employed in the proof of Theorem 1.1. In particular, the proof of Theorem 1.2 uses in the very essential way the solution of the Restricted Burnside Problem [21, 22] while the proof of Theorem 1.1 is based on more simple techniques. Some explanation of this phenomenon can be found in the study of automorphisms of Lie algebras.

By the Kreknin Theorem [8] a Lie algebra that admits a fixed-point-free automorphism of finite order n is soluble with n -bounded derived length. On the other hand, a Lie algebra that admits a non-cyclic fixed-point-free group of automorphisms can be unsoluble or soluble with arbitrarily large derived length (see examples in [4, p. 149–150]). It is the necessity to work with Lie algebras of unbounded derived length that accounts for the complexity of the proof of Theorem 1.2.

The present paper is a natural continuation of [19]. Here we study in more detail questions about the exponent of a finite group admitting a fixed-point-free four-group of automorphisms. In the first result that we would like to mention we consider groups acted on by S_4 , the symmetric group on 4 symbols. In what follows we denote by V the maximal normal 2-subgroup of S_4 . Of course V is the non-cyclic group of order 4.

THEOREM 1.3. *Let A be isomorphic with S_4 and let α be an involution in $A \setminus V$. Suppose that A acts on a finite group G in such a manner that $C_G(V) = 1$ and $C_G(\alpha)$ has exponent e . Then the exponent of G is bounded in terms of e only.*

Remark that the above result does not require the coprimeness assumption. It is interesting and somewhat unusual that Theorem 1.3 involves only a hypothesis on the exponent of $C_G(\alpha)$ rather than the centralizer of the subgroup of order three as in Theorem 1.2. It is well-known that the Sylow 2-subgroup of S_4 is isomorphic with D_8 , the dihedral group of order 8. When studying the action of D_8 on G satisfying the conditions similar to the ones in Theorem 1.3 we discovered a new phenomenon – such an action actually has strong impact on the exponent of G' , the derived group of G . Write D_8 as a product $V\langle\alpha\rangle$, where V is a four-group and α an involution.

THEOREM 1.4. *Let A be isomorphic with D_8 and suppose that A acts on a finite group G in such a manner that $C_G(V) = 1$ and $C_G(\alpha)$ has exponent e . Then the exponent of G' is bounded in terms of e only.*

The proofs of both Theorem 1.3 and Theorem 1.4 use the solution of the Restricted Burnside Problem. Another important tool used in the proof of the above results is the theorem that the exponent of a finite group acted on by a non-cyclic abelian group A is bounded in terms of $|A|$ and the exponents of $C_G(a)$, where $a \in A \setminus \{1\}$ [7]. As a part of the proof of Theorem 1.4 we obtained the following result that seems to be of independent interest.

THEOREM 1.5. *Let G be a finite group acted on by the four-group V in such a manner that $C_G(V) = 1$. Suppose that the centralizers $C_G(v_1)$*

and $C_G(v_2)$ of two involutions $v_1, v_2 \in V$ have exponent e . Then the exponent of G' is bounded in terms of e only.

In view of the above results a number of questions about the exponent of a finite group with automorphisms can be asked. In particular, it would be interesting to see if similar results hold in the situation where V is an elementary abelian p -group for an odd prime p .

Throughout the paper we use the expression “ (m, n) -bounded” for “bounded above in terms of m, n only”.

2. A criterion of nilpotency for Lie algebras

In this section we will describe some key Lie-theoretic tools required for the proofs of the main results. In particular we will establish a sufficient condition for a Lie algebra with automorphisms to be nilpotent. Though the hypotheses of Proposition 2.3 below may look bizarre, the proposition is sufficient (and perhaps even necessary) for the group-theoretic applications that will be obtained in Section 5. In what follows the term “Lie algebra” means a Lie algebra over some commutative ring with unity in which 2 is invertible. If $X \subseteq L$ is a subset of a Lie algebra L , we denote by $\langle X \rangle$ the subalgebra generated by X . By a commutator of weight 1 in elements of X we mean just any element of X . We define inductively commutators in X of weight $w \geq 2$ as elements of the form $[x, y]$, where x and y are commutators in X of weight w_1 and w_2 respectively such that $w_1 + w_2 = w$. As usual, $Z(L)$ and $\gamma_i(L)$ denote the center and the i th term of the lower central series of L , respectively. The centralizer $C_L(S)$ of a subset S is the subalgebra comprised of all elements $x \in L$ such that $[S, x] = 0$. If a group A acts by automorphisms on L , we denote by $C_L(A)$ the fixed subalgebra of L . Recall that an element a of a Lie algebra L is called ad-nilpotent if there exists a positive integer m such that $[x, \underbrace{a, \dots, a}_m] = 0$ for all

$x \in L$. If m is the least integer with the above property, then we say that a is ad-nilpotent of index m .

Let $A = V\langle\alpha\rangle$ be the dihedral group of order 8 with $V = \{1, v_1, v_2, v_3\}$ being a four-group and α an involution such that $v_1^\alpha = v_2$. Let A act on a Lie algebra L in such a way that $C_L(V) = 0$. For $i = 1, 2, 3$ set $L_i = C_L(v_i)$. Then we have $L = \bigoplus_{1 \leq i \leq 3} L_i$, where L_i are abelian subalgebras with the property that

$$[L_1, L_2] \leq L_3, \quad [L_2, L_3] \leq L_1, \quad [L_3, L_1] \leq L_2.$$

Of course, the subalgebras L_i are V -invariant and v_j acts on L_i by taking every element $x \in L_i$ to $-x$ whenever $i \neq j$. Moreover

$$L_1^\alpha = L_2 \quad \text{and} \quad L_3^\alpha = L_3.$$

If $X \subseteq L$ is a subset of L , we denote by $I(X)$ the ideal of L generated by X and by $ID(X)$ the minimal A -invariant ideal of L containing X .

The next two lemmas are taken from [16].

LEMMA 2.1. ([16, Lemma 1.2]) *Suppose that $[a, b] = 0$, where $a \in L_i$, $b \in L_j$ for some $i \neq j$. Then $I([b, L_i]) \leq C_L(a)$.*

LEMMA 2.2. ([16, Proposition 1.1]) *Let $a \in L_1 \cup L_2 \cup L_3$ and suppose that a is ad-nilpotent of index m . Then $I(a)$ is nilpotent of class at most $2m - 1$.*

In the sequel we use the fact that the quotient $L/ID(X)$ naturally satisfies all the necessary assumptions without explicitly mentioning it. More generally, this applies to any quotient over an A -invariant ideal. Certainly $\gamma_r(L)$ and $C_L(\gamma_r(L))$ are always A -invariant.

We will write L_α for $C_L(\alpha)$. Given a set $X \subseteq L$ such that $L = \langle X \rangle$, an element of L is said to be homogeneous (of weight w) with respect to the generating set X if it can be written as a homogeneous Lie polynomial (of degree w) in elements of X .

Our main result on nilpotency of Lie algebras is as follows.

PROPOSITION 2.3. *Let $A = V\langle\alpha\rangle$ be the dihedral group of order 8 acting on a Lie algebra L in such a way that $C_L(V) = 0$. Assume that there exists $x_1 \in L_1$ such that $L = \langle x_1, x_1^\alpha \rangle$. Moreover, for the generating set $\{x_1, x_1^\alpha\}$ there exist positive integers m and n such that every homogeneous element contained in L_α is ad-nilpotent in L of index at most m and every pair of homogeneous elements contained in L_α generates a subalgebra that is nilpotent of class at most n . Then L is nilpotent of (m, n) -bounded class.*

First we establish the following related result.

PROPOSITION 2.4. *Assume the hypothesis of Proposition 2.3 and let L be soluble with derived length k . Then L is nilpotent of (k, m, n) -bounded class.*

PROOF. By the main result in [18], L' is nilpotent of k -bounded class. Using the Lie algebra analogue of Hall's criterion of nilpotency [2], we can assume that L is metabelian. Put $x_2 = x_1^\alpha$, $x = x_1 + x_2$ and $y = x_1 - x_2$. It is clear that $L = \langle x, y \rangle$. Since x is a homogeneous element contained in L_α , it follows that x is ad-nilpotent in L of index at most m . We also notice that $y = x^{v_1}$. Therefore y is ad-nilpotent

in L of index at most m , as well. Thus, L is a metabelian Lie algebra generated by two ad-nilpotent elements of index at most m . It follows that L is nilpotent of (k, m, n) -bounded class. \square

For every $x \in L$ we write $x = x_1 + x_2 + x_3$, where $x_i \in L_i$. We will require the following elementary lemma.

LEMMA 2.5. *Suppose that $y \in L_i$ for some i and assume that $x \in C_L(y)$. Then also $x_1, x_2, x_3 \in C_L(y)$.*

PROOF. We notice that the 1-dimensional subspace $\langle y \rangle$ is A -invariant. Therefore the centralizer $C_L(y)$ is A -invariant as well. Hence $C_L(y)$ is the direct sum of the subspaces $C_L(y) \cap L_k$ for $k = 1, 2, 3$. The lemma follows. \square

PROOF OF PROPOSITION 2.3. Set $x = x_1 + x_2$. It is clear that $x \in L_\alpha$. By the hypothesis, x is ad-nilpotent in L of index at most m . Let k be the minimal number such that

$$[x_1 - x_2, \underbrace{x, \dots, x}_k] = 0.$$

The proposition will be proved by induction on k . We know that $k \leq m$. If $k = 1$, then, since 2 is invertible in the ground ring of L , it follows that x_1 and x_2 commute and so L is abelian. Hence, we can assume that $k \geq 2$. In what follows we will call an element $y \in L$ *critical* to mean that $L/ID(y)$ is nilpotent of (m, n) -bounded class. Thus, the induction hypothesis is that the elements $[x_1 - x_2, \underbrace{x, \dots, x}_i]$ are critical

for all $i \leq k - 1$. Set $t = [x_1 - x_2, \underbrace{x, \dots, x}_{k-1}]$. If $k - 1$ is odd, then $t \in L_3$.

Taking into account that t lies in the centralizer of x and using Lemma 2.5, we conclude that t commutes with x_1 and x_2 . Hence, t is a critical element that belongs to $Z(L)$ and so the result follows.

Therefore we assume that $k - 1$ is even and so $t \in L_1 + L_2$. We remark that $t \notin L_\alpha$ because it has the form $l - l^\alpha$ for suitable $l \in L$.

Consider the elements $d = t_1 + t_1^\alpha$, $l = t - d$ and $g = [x, d]$. It is easy to see that $d \in L_\alpha$, $0 \neq l \in L_2$ and $g \in L_3$.

Suppose first that $g = 0$. Then l commutes with x . It follows from Lemma 2.5 that $l \in Z(L)$. We remark that l is a critical element. Hence, the result follows.

Therefore we can assume that $g \neq 0$. We will use the number of distinct commutators in x and d as the second induction parameter. By the hypothesis this number is n -bounded. The induction hypothesis will be that every non-zero commutator in x and d is a critical element.

Choose a commutator z in x and d such that $0 \neq z \in Z(\langle x, d \rangle)$. It is clear that z is a critical element. Since both x and d lie in $L_1 + L_2$, every commutator in x and d lies either in L_3 or in $L_1 + L_2$. If $z \in L_3$, using the fact that $[z, x] = 0$ we deduce that z belongs to $Z(L)$ and so the result follows. Suppose that $z = z_1 + z_2 \in L_1 + L_2$. Since $z \in L_\alpha$, it follows that $z_2 = z_1^\alpha$. Because $g \in L_3$ and $[g, z] = 0$, Lemma 2.5 implies that g commutes with both z_1 and z_2 . Moreover, taking into account that $[x_1, x_2] \in L_3$ and using Lemma 2.1 we conclude that g commutes with both ideals $I([x_1, x_2, z_1])$ and $I([x_1, x_2, z_2])$.

We remark that $[x_1, x_2, z_1] - [x_1, x_2, z_2] \in L_\alpha$ and so this element is ad-nilpotent of index at most m . Since

$$([x_1, x_2, z_1] - [x_1, x_2, z_2])^{v_2} = [x_1, x_2, z_1] + [x_1, x_2, z_2],$$

we conclude that also $[x_1, x_2, z_1] + [x_1, x_2, z_2]$ is ad-nilpotent of index at most m .

Set $N = I([x_1, x_2, z_1]) \cap I([x_1, x_2, z_2])$ and

$$J = I([x_1, x_2, z_1]) + I([x_1, x_2, z_2]) = ID([x_1, x_2, z_1]).$$

Suppose that $N = 0$. In this case $[x_1, x_2, z_1]$ commutes with $[x_1, x_2, z_2]$. Since 2 is invertible in the ground ring of L and both $[x_1, x_2, z_1] + [x_1, x_2, z_2]$ and $[x_1, x_2, z_1] - [x_1, x_2, z_2]$ are ad-nilpotent of index at most m , we deduce that $[x_1, x_2, z_1]$ and $[x_1, x_2, z_2]$ are ad-nilpotent of index at most $2m - 1$.

Lemma 2.2 shows that both ideals $I([x_1, x_2, z_1])$ and $I([x_1, x_2, z_2])$ are nilpotent of index at most $4m - 3$ and so the ideal J is nilpotent of bounded class. Let us pass to the quotient L/J (we use of course that J is A -invariant). For simplicity we just assume that $J = 0$. Thus both z_1 and z_2 commute with $[x_1, x_2]$. By Lemma 2.1 z_1 commutes with $I([x_1, x_2, x_1])$ and z_2 commutes with $I([x_1, x_2, x_2])$. Therefore z commutes with the intersection $M = I([x_1, x_2, x_1]) \cap I([x_1, x_2, x_2])$.

If $M = 0$, then $[x_1, x_2, x_1]$ commutes with $[x_1, x_2, x_1]^\alpha = [x_2, x_1, x_2]$. With the same argument of above, we deduce that $ID([x_1, x_2, x_1])$ is nilpotent of bounded class. Passing to the quotient $L/ID([x_1, x_2, x_1])$ we can assume that $[x_1, x_2]$ commutes with both x_1 and x_2 . In this case L is nilpotent of class at most 2. Hence, $L/ID([x_1, x_2, x_1])$ is nilpotent of class at most 2. In particular we have shown that if $M = 0$ then L is soluble with bounded derived length. Proposition 2.4 yields that if $M = 0$ then L is nilpotent of bounded class. Hence, M contains $\gamma_i(L)$ for some bounded i . Using the fact that z is a critical element that centralizes M we now deduce that the algebra L is soluble with bounded derived length. Proposition 2.4 now implies that L is nilpotent

with bounded class. Recall that we have assumed that $N = 0$. Now we will drop this assumption.

Since N is A -invariant, we can now conclude that N contains $\gamma_j(L)$ for some bounded j . Recall that g commutes with both ideals $I([x_1, x_2, z_1])$ and $I([x_1, x_2, z_2])$. In particular, it follows that g commutes with N . Therefore we found that g is a critical element commuting with $\gamma_j(L)$ for some bounded j . We conclude that L is soluble with bounded derived length. Another application of Proposition 2.4 completes the proof. \square

3. Associating Lie algebra to a group

The proofs of the main results of this paper are based on the so called Lie methods. We will now describe the construction that associates with any group a Lie algebra over the field with p elements. This section does not contain new results and is given here only for the reader's convenience.

Let G be a group and p a prime. We set

$$D_i = D_i(G) = \prod_{jp^k \geq i} \gamma_j(G)^{p^k}.$$

The subgroups D_i form the Jennings–Zassenhaus filtration

$$G = D_1 \geq D_2 \geq \cdots$$

of the group G . The series satisfies the inclusions $[D_i, D_j] \leq D_{i+j}$ and $D_i^p \leq D_{pi}$ for all i, j . These properties make it possible to construct a Lie algebra $DL(G)$ over \mathbb{F}_p , the field with p elements. Namely, consider the quotients $K_i = D_i/D_{i+1}$ as linear spaces over \mathbb{F}_p , and let $DL(G)$ be the direct sum of these spaces. Commutation in G induces a binary operation $[\cdot, \cdot]$ in $DL(G)$. For elements $xD_{i+1} \in K_i$ and $yD_{j+1} \in K_j$ the operation is defined by

$$[xD_{i+1}, yD_{j+1}] = [x, y]D_{i+j+1} \in K_{i+j}$$

and extended to arbitrary elements of $DL(G)$ by linearity. It is easy to check that the operation is well-defined and that $DL(G)$ with the operations $+$ and $[\cdot, \cdot]$ is a Lie algebra over \mathbb{F}_p .

For any $x \in D_i \setminus D_{i+1}$ let \bar{x} denote the element xD_{i+1} of $DL(G)$.

LEMMA 3.1 (Lazard [10]). *For any $x \in G$ we have $(\text{ad } \bar{x})^p = \text{ad } \bar{x}^p$. Consequently, if x is of finite order p^t , then \bar{x} is ad-nilpotent of index at most p^t .*

Denote by $L_p(G)$ the subalgebra of $DL(G)$ generated by $K_1 = D_1/D_2$. The following lemma goes back to Lazard [11]; in the present form it can be found, for example, in [7].

LEMMA 3.2. *Suppose that G is a d -generated finite p -group such that the Lie algebra $L_p(G)$ is nilpotent of class c . Then G has a powerful characteristic subgroup of (p, c, d) -bounded index.*

Recall that powerful p -groups were introduced by Lubotzky and Mann in [12]: a finite p -group G is *powerful* if $G^p \geq [G, G]$ for $p \neq 2$ (or $G^4 \geq [G, G]$ for $p = 2$). These groups have many nice properties, so that often a problem becomes much easier once it is reduced to the case of powerful p -groups. The above lemma is quite useful as it allows us to perform such a reduction. Precisely, we will require the property that if $G = \langle g_1, \dots, g_d \rangle$ is a powerful p -group and $e = p^k$, then $G^e = \langle g_1^e, \dots, g_d^e \rangle$. Using this it is not really difficult to prove Theorems 1.5, 1.4 and 1.3 in the particular case where G is a powerful p -group. Thus, we see the general idea of proofs of the main results – the reduction to the powerful p -groups will be performed via Lemma 3.2 while the nilpotency of the corresponding Lie algebras will be established through Proposition 2.3.

4. Proof of Theorem 1.5

We start this section with citing some useful results about finite groups admitting a fixed-point-free four-group of automorphisms. Without further references we use the well-known fact that if a finite group A acts coprimely on a finite group G and N is a normal A -invariant subgroup of G , then $C_{G/N}(A) = C_G(A)N/N$.

Let G be a finite group and $V = \{1, v_1, v_2, v_3\}$ the non-cyclic group of order 4 acting fixed-point-freely on G . Then G has odd order. Put $G_i = C_G(v_i)$ for $i \in \{1, 2, 3\}$. The proofs of the next few lemmas can be found in [15].

LEMMA 4.1. *Each G_i is abelian and if $i \neq j$ then v_j acts on G_i by the rule $x^{v_j} = x^{-1}$ for each $x \in G_i$; $i, j \in \{1, 2, 3\}$. It follows that every subgroup generated by a subset of $G_1 \cup G_2 \cup G_3$ is V -invariant.*

LEMMA 4.2. *Let x be an element of G such that $x^{v_i} = x^{-1}$ for some i . Suppose that $x \in S$, where S is some V -invariant subgroup of G . Then there exists a unique pair of elements $y \in G_j \cap S$ and $t \in G_k \cap S$ such that $x = yty$ and $\{i, j, k\} = \{1, 2, 3\}$.*

LEMMA 4.3. $G = G_1 G_2 G_3$.

LEMMA 4.4. $G' = \langle G_1, G_2 \rangle \cap \langle G_2, G_3 \rangle \cap \langle G_3, G_1 \rangle$. In particular, the subgroups $\langle G_i, G_j \rangle$ are normal and contain G' .

For any $x \in G_i$ and $y \in G_j$ with $i \neq j$ it is clear that v_i sends y^x to the inverse. Therefore Lemma 4.2 guarantees that there exists a

unique pair $(s, t) \in G_k \times G_j$ such that $y^x = sts$ and $\{i, j, k\} = \{1, 2, 3\}$. Thus we can define $x * y = s$. According to the same lemma, if y^x is an element of a V -invariant subgroup S , then $x * y \in S$. The next lemma is taken from [17].

LEMMA 4.5. *Let $x \in G_i, y \in G_j$ and $H = \langle x, y \rangle$. Then $\langle (x * y)^H \rangle = H'$.*

Let us define

$$R_1 = \langle a * b \mid a \in G_2, b \in G_3 \rangle,$$

$$R_2 = \langle a * b \mid a \in G_1, b \in G_3 \rangle,$$

$$R_3 = \langle a * b \mid a \in G_1, b \in G_2 \rangle$$

and

$$T_1 = \langle b * a \mid a \in G_2, b \in G_3 \rangle,$$

$$T_2 = \langle b * a \mid a \in G_1, b \in G_3 \rangle,$$

$$T_3 = \langle b * a \mid a \in G_1, b \in G_2 \rangle.$$

LEMMA 4.6. *If G is nilpotent, we have $R_i = T_i$ for $i = 1, 2, 3$.*

PROOF. Let G be a counterexample of minimal order. If N is any normal V -invariant subgroup of G , by induction we have $T_i N = R_i N$. Let $Z = Z(G)$. If Z contains a nontrivial element of the form $a * b$ for some $a \in G_i, b \in G_j$, we put $N = \langle a * b \rangle$. It is clear that N is a normal V -invariant subgroup contained either in some T_i or in some R_i . Since it is central, Lemma 4.5 shows that actually N contains both $a * b$ and $b * a$. Hence $N \leq R_i \cap T_i$ and so $R_i = T_i$.

We therefore assume that Z contains no nontrivial elements of the form $a * b$. Let $K = Z_2(G)$ be the second term of the upper central series of G and set $K_i = K \cap G_i$ for $i = 1, 2, 3$. Choose arbitrarily $a \in G_i$ and $b \in K_j$. Then, because of Lemma 4.5, it follows that $a * b \in Z(G)$. Therefore we conclude that $a * b = 1$ and so by Lemma 4.5 $[a, b] = 1$. This happens for every choice of $a \in G_i$ and $b \in K_j$. Hence $K = Z(G)$ and G is abelian, a contradiction. \square

We will also require the following result, whose proof is similar to that of Lemma 4.6.

LEMMA 4.7. *If G is nilpotent, then $G_i \cap G' = R_i$ for $i = 1, 2, 3$.*

PROOF. Let G be a counterexample of minimal order. Set $K = G'$ and $K_i = K \cap G_i$ for $i = 1, 2, 3$. If N is any normal V -invariant subgroup of G , by induction we have $K_i N = R_i N$. Let $Z = Z(G)$. If Z contains a nontrivial element of the form $a * b$ for some $a \in G_i, b \in G_j$, we put $N = \langle a * b \rangle$. It is clear that N is a normal V -invariant subgroup. By Lemma 4.6 N contained in $R_1 \cup R_2 \cup R_3$ and in view of the above

assumption we get a contradiction. Thus, assume that Z contains no nontrivial elements of the form $a * b$.

Let $T = Z_2(G)$ be the second term of the upper central series of G and set $T_i = T \cap G_i$ for $i = 1, 2, 3$. Choose arbitrarily $a \in G_i$ and $b \in T_j$. Then, because of Lemma 4.5, it follows that $a * b \in Z(G)$. Therefore we conclude that $a * b = 1$ and so by Lemma 4.5 $[a, b] = 1$. This happens for every choice of $a \in G_i$ and $b \in T_j$. Hence $T = Z(G)$ and G is abelian, a contradiction. \square

Another useful fact is provided by the next lemma.

LEMMA 4.8. *Suppose that N is a normal subgroup of G such that $N \leq G_3$. Then $N \leq Z(G)$.*

PROOF. Choose $b \in N$ and $a \in G_1$ (or $a \in G_2$). Since $b^a \in G_3$, it follows that $a * b = 1$ and so by Lemma 4.5 $[a, b] = 1$. Thus an arbitrary element of N commutes with an arbitrary element of $G_1 \cup G_2$. Using that G_3 is abelian and $N \leq G_3$ we conclude that $N \leq Z(G)$. \square

Before embarking on the proof of Theorem 1.5 we quote the main result of [7] that plays a crucial role in subsequent arguments.

THEOREM 4.9. *Let q be a prime, e a positive integer and A an elementary abelian group of order q^2 . Suppose that A acts as a coprime group of automorphisms on a finite group G and assume that $C_G(a)$ has exponent dividing e for each $a \in A \setminus \{1\}$. Then the exponent of G is $\{e, q\}$ -bounded.*

PROOF OF THEOREM 1.5. Let $K = G'$. Put $G_i = C_G(v_i)$ and $K_i = K \cap G_i$ for $i \in \{1, 2, 3\}$. By Lemma 4.3, we have $G = G_1 G_2 G_3$ and $K = K_1 K_2 K_3$. By the hypothesis, the centralizers G_1 and G_2 have exponent e and therefore the exponents of K_1 and K_2 divide e . In view of Theorem 4.9 it is sufficient to prove that K_3 has e -bounded exponent.

Suppose first that G is a p -group. Lemma 4.7 tells us that K_3 is generated by elements of the form $a * b$ for $a \in G_1$ and $b \in G_2$. Since K_3 is abelian, it is sufficient to show that $a * b$ has e -bounded order for every $a \in G_1$ and $b \in G_2$. Thus, choose $a \in G_1$ and $b \in G_2$ and without loss of generality we can assume that $G = \langle a, b \rangle$. Let $L = L_p(G)$. Combining Lemma 3.1 and Lemma 2.2, we deduce that L is nilpotent of e -bounded class. Therefore by Lemma 3.2 G has a powerful characteristic subgroup H of e -bounded index. Since H is V -invariant, it is generated by the centralizers $H \cap G_i$. Hence, we can choose generators g_1, \dots, g_d of H such that $g_1, \dots, g_d \in G_1 \cup G_2 \cup G_3$. Since $G_1^e = G_2^e = 1$ and $H^e = \langle g_1^e, \dots, g_d^e \rangle$, we conclude that $H^e \leq$

G_3 . Thus, Lemma 4.8 tells us that $H^e \leq Z(G)$ and so $H/(Z(G) \cap H)$ has e -bounded exponent. Therefore, according to the solution of the restricted Burnside problem, $G/Z(G)$ has e -bounded order and using the Schur Theorem [14, p. 102] we conclude that G' has e -bounded order. In particular the order of $a * b$ is e -bounded. Thus, in the case where G is a p -group the theorem is valid. This extends immediately to the case where G is nilpotent.

In general a finite group admitting a fixed-point-free four-group of automorphisms need not be nilpotent. However it is known that the derived group of such a group is nilpotent [1, Theorem 10.5.3]. Therefore we deduce now that G'' has e -bounded exponent. Passing to the quotient G/G'' , we can assume that G is metabelian. Because it is sufficient to bound the exponent of K_3 , we can also assume that $G = \langle G_1, G_2 \rangle$. Since K is generated by the centralizers K_i and $G_1^e = G_2^e = 1$, it follows that $K^e \leq G_3$. Thus, Lemma 4.8 tells us that $K^e \leq Z(G)$, and hence $K/(Z(G) \cap K)$ has e -bounded exponent. Since $G = \langle G_1, G_2 \rangle$, it is clear that the exponent of G/K divides e and so $G/Z(G)$ has e -bounded exponent. If $x, y \in G$, we see that $\langle x, y \rangle / Z(\langle x, y \rangle)$ has e -bounded order. Therefore by Schur's theorem $|\langle x, y \rangle'|$ is e -bounded, as well. In particular, the order of every commutator $[x, y]$ is e -bounded. Since G is metabelian, it follows that K has e -bounded exponent. The proof is now complete. \square

5. Main results

We will now prove Theorem 1.4 and Theorem 1.3.

PROOF OF THEOREM 1.4. Recall that the group $A = V\langle \alpha \rangle$, isomorphic with D_8 , acts on a finite group G . We use the notation introduced in the previous parts of the paper and assume that $G_1^\alpha = G_2$ and G_3 is α -invariant.

Let H be any A -invariant subgroup of G and N the minimal A -invariant normal subgroup of H containing $C_H(\alpha)$. Then α is fixed-point-free on the quotient H/N . It follows that A induces an abelian group of automorphisms of H/N and in particular, since $v_3 \in A'$, it follows that v_3 must act trivially on H/N . The conclusion is that $H = NC_H(v_3)$ and a similar decomposition holds for all A -invariant subgroups of G and all A -invariant quotients.

In view of Theorem 1.5 the exponent of G' is bounded by some number that depends only on the exponents of G_1 and G_2 . Since $G_1^\alpha = G_2$, it is sufficient to show that an arbitrary element x_1 of G_1 has e -bounded order. Put $x_1^\alpha = x_2$. The subgroup $\langle x_1, x_2 \rangle$ is A -invariant so without loss of generality we can assume that $G = \langle x_1, x_2 \rangle$.

We remark that v_3 has no fixed-points in G/G' and recall the formula $H = NC_H(v_3)$. It follows that G/G' is generated by subgroups of the form $(C_{G/G'}(\alpha))^v$ where v ranges through V . Hence, G/G' is generated by elements of order dividing e . Since the rank of G/G' is at most two, we conclude that the order of G/G' is e -bounded.

Suppose first that G is a p -group and let $L = L_p(G)$. The group A naturally acts on L and we will show that L satisfies all the hypothesis of Proposition 2.3. Indeed, the fact that every homogeneous element contained in L_α is ad-nilpotent in L of index at most e follows from Lazard's Lemma 3.1, while the existence of an e -bounded number n such that every pair of homogeneous elements in L_α generates a subalgebra that is nilpotent of class at most n is immediate from Zelmanov's solution of the restricted Burnside problem [21, 22]. Set $L_i = C_L(v_i)$ for each $i \in \{1, 2, 3\}$. Since every L_i admits a fixed-point-free involutory automorphism, it follows that every L_i is abelian. Thus, Proposition 2.3 tells us that L is nilpotent with e -bounded class. According to Lemma 3.2 G has a powerful characteristic subgroup H of e -bounded index. Using the formula $H = NC_H(v_3)$, we can choose generators g_1, \dots, g_d of H each of which belongs either to a conjugate of $C_H(\alpha)$ or to $C_H(v_3)$. Taking into account that $C_H(\alpha)$ has exponent e and that $H^e = \langle g_1^e, \dots, g_d^e \rangle$, we derive that $H^e \leq G_3$. It follows now from Lemma 4.8 that $H/(Z(G) \cap H)$ has e -bounded exponent, and hence $G/Z(G)$ has e -bounded order. Using the Schur Theorem we conclude that G' has e -bounded order. We know that the order of G/G' is e -bounded, as well. So $G = \langle x_1, x_2 \rangle$ has e -bounded order, and hence x_1 has e -bounded order, too. Since x_1 was chosen in G_1 arbitrarily, we conclude that G_1 has bounded exponent and hence the same for G_2 . This proves the theorem in the case where G is a p -group. Of course, from this the case where G is nilpotent is straightforward.

Let us now deal with the case where G is not necessarily nilpotent. We know from [1, 10.5.3] that G' is nilpotent. Thus, we deduce from the previous paragraph that G'' has e -bounded exponent. Passing to the quotient G/G'' , we can assume that G is metabelian. Let $M = G'$. Since $M = M_1 M_2 M_3$ where $M_i = M \cap G_i$, we see that M is generated by M_3 and some elements of order e . Therefore $M^e \leq G_3$. According to Lemma 4.8 it follows that $M^e \leq Z(G)$. As in the proof of Theorem 1.5 we conclude that G' has e -bounded order. Combining this with the fact that the order of G/G' is likewise e -bounded, the theorem follows. \square

PROOF OF THEOREM 1.3. Since the group $V\langle\alpha\rangle$ is isomorphic with D_8 , we can use all the information that we obtained in the proof of Theorem 1.4. It follows that the exponent of G_1 is e -bounded. Of course

in the group S_4 all involutions contained in V are conjugate so we conclude that G_1 , G_2 and G_3 have the same exponent. Now Theorem 4.9 tells us that the exponent of G is e -bounded, as required. \square

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